Quantum Bit Error Estimation Based on the Syndrome of a Linear Code

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SUMMARY

We derive and analyze a Maximum Likelihood (ML) estimator for the quantum bit error rate (QBER). The estimator is based on Low-Density Parity-Check (LDPC) codes. Bob takes as input only his raw key and the syndrome he has received from Alice. We focus our analysis [1] on check-regular LDPC codes where every row of the parity-check matrix has constant weight but briefly address the check-irregular case with non-constant weights as well. We obtain a quite accurate estimator that can be used for two tasks in QKD: as an improvement over the sampling estimator (which compares sets of individual bits), and to improve the efficiency in interactive reconciliation protocols.

1 RECONCILIATION WITH LOW-DENSITY PARITY-CHECK (LDPC) CODES

- One way to define a binary linear error correcting code is by means of its parity-check matrix H: The null-space of the parity-check matrix defines the set of all codewords: $\mathcal{C} = \left\{\boldsymbol{x} \in \{0,1\}^n : \boldsymbol{x}\mathrm{H}^T = \boldsymbol{0}\right\}$.
- If H is sparse the code is called *Low-Density Parity-Check (LDPC) code* [2].
- Codes with constant weight d (called check degree) in each row are *check-regular*.
- An important application is reconciliation of data in quantum crypto: Assume Alice and Bob have obtained correlated vectors, $\boldsymbol{x}_A$ and $\boldsymbol{x}_B = \boldsymbol{x}_A \oplus \boldsymbol{e}$, resp., where $\boldsymbol{e}$ is the errorword (of low weight).
  Then Alice calculates the syndrome $\boldsymbol{S}_A := \boldsymbol{x}_A\mathrm{H}^T$ of her vector $\boldsymbol{x}_A$ and an LDPC code with parity-check matrix H and sends $\boldsymbol{S}_A$ on an error-free channel to Bob. If the quantum bit error rate has not been too large, Bob can reconstruct $\boldsymbol{x}_A$ from $\boldsymbol{x}_B$ and $\boldsymbol{S}_A$.
- But, can we reuse the syndrome for further purposes?

2 ERROR ESTIMATION WITH LDPC CODES

Yes, Bob can :) estimate the quantum bit error rate prior to decoding!
We model the errors from the quantum channel, i.e. the individual bits of $\boldsymbol{e}$ as iid: $\Pr\{e_i = 1\} = \rho$, where ρ denotes the quantum bit error rate (QBER).

- Bob performs the calculation

$$\boldsymbol{S} := \boldsymbol{e}\mathrm{H}^T = (\boldsymbol{x}_A \oplus \boldsymbol{x}_B)\,\mathrm{H}^T = \boldsymbol{S}_A \oplus \boldsymbol{x}_B\mathrm{H}^T.$$

The individual bits of the syndrome $\boldsymbol{S}$ can be well approximated to also be i.i.d. The approximation consists in neglecting the (weak) correlation between syndrome bits that sum over a common data bit, x_i.

- With this approximation the probability q that a syndrome bit is one is [2]

$$q = f_d(\rho) := \sum_{\substack{1 \le i \le d \\ i \text{ odd}}} \binom{d}{i} \rho^i (1-\rho)^{d-i} = \frac{1-(1-2\rho)^d}{2}. \tag{1}$$

3 DERIVATION OF THE MAXIMUM LIKELIHOOD ESTIMATOR FOR THE QBER

Let m denote the length of $\boldsymbol{S}$, and $W = \mathrm{wt}\{\boldsymbol{S}\}$ denote the Hamming weight of $\boldsymbol{S}$. The syndrome weight W is a binomially distributed random variable, i.e.,

$$\Pr\{W = w\} = f_{binom}(w; m, q) := \binom{m}{w} q^w (1-q)^{m-w}, \tag{2}$$

and the maximum likelihood (ML) estimate for ρ given a syndrome weight w is

$$\hat{\rho}(w) = \arg\max_{\rho'} \left\{ f_{binom}(w; m, f_d(\rho')) \right\}, \tag{3}$$

which can be solved analytically. Equivalently, one can take the ML estimator for q

$$\hat{q}(w) = \frac{w}{m}, \tag{4}$$

and use it with (1) to obtain the estimate $\hat{\rho}$. Both approaches give the same result:

- The final estimator in closed form is

$$\hat{\rho}(w) = \begin{cases} \frac{1-\left(1-2\frac{w}{m}\right)^{\frac{1}{d}}}{2} & ; \frac{w}{m} \le 1/2 \\ \frac{1}{2} & ; \frac{w}{m} > 1/2 \end{cases}. \tag{5}$$

This approach can be generalized to check-irregular LDPC codes with different check degrees by replacing the binomial distribution in (2) with a multinomial distribution.

4 PROPERTIES OF THE MAXIMUM LIKELIHOOD ESTIMATOR

- Mean

$$\mu(d,\rho,m) = \mathbb{E}_W[\hat{\rho}(W)] = \frac{1}{2} - \frac{1}{2}\sum_{w=0}^{\lfloor m/2 \rfloor} f_{binom}(w; m, f_d(\rho)) \left(1 - 2\frac{w}{m}\right)^{\frac{1}{d}}, \tag{6}$$

- Bias

$$B(d,\rho,m) = \mu(d,\rho,m) - \rho. \tag{7}$$

- Mean squared error (MSE)

$$\begin{aligned} \mathrm{MSE}(d,\rho,m) = & \mathbb{E}_W\left[(\hat{\rho}(W)-\rho)^2\right] = \frac{1}{4} - 2\rho\mu(d,\rho,m) + \rho^2 \\ & + \frac{1}{4}\sum_{w=0}^{\lfloor m/2 \rfloor} f_{binom}(w; m, f_d(\rho)) \left(\left(1-2\frac{w}{m}\right)^{\frac{2}{d}} - 2\left(1-2\frac{w}{m}\right)^{\frac{1}{d}} \right). \end{aligned} \tag{8}$$

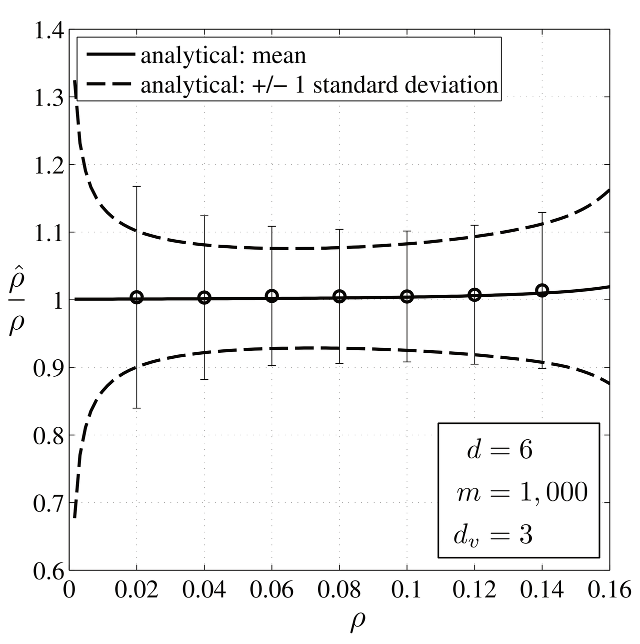
- Cramér-Rao Lower Bound
  The mean squared error of any biased estimator is lower bounded by

$$\mathrm{MSE}(d,\rho,m) \ge \frac{\left(\frac{\partial}{\partial\rho}\mu(d,\rho,m)\right)^2}{\mathcal{I}(\rho)} + B^2(d,\rho,m), \tag{9}$$

  where $\mathcal{I}(\rho)$ is the Fisher information that the syndrome $\boldsymbol{S}$ carries about ρ:

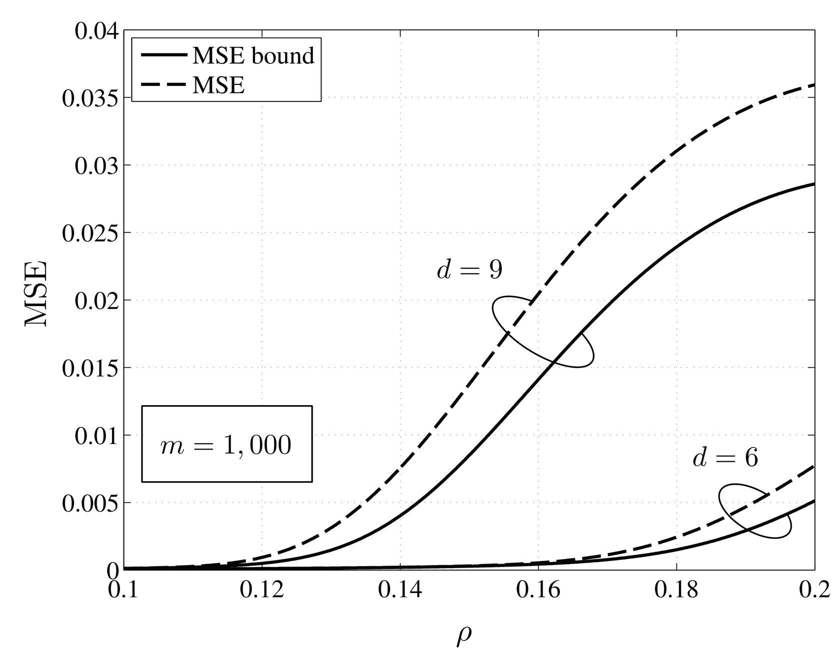
$$\mathcal{I}(\rho) = -\mathbb{E}_{\boldsymbol{S}}\left[\frac{\partial^2}{\partial\rho^2}\log\Pr\{\boldsymbol{S};\rho\}\right] = \frac{4md^2(1-2\rho)^{2d-2}}{1-(1-2\rho)^{2d}}. \tag{10}$$

5 NORMALIZED MEAN AND STANDARD DEVIATION



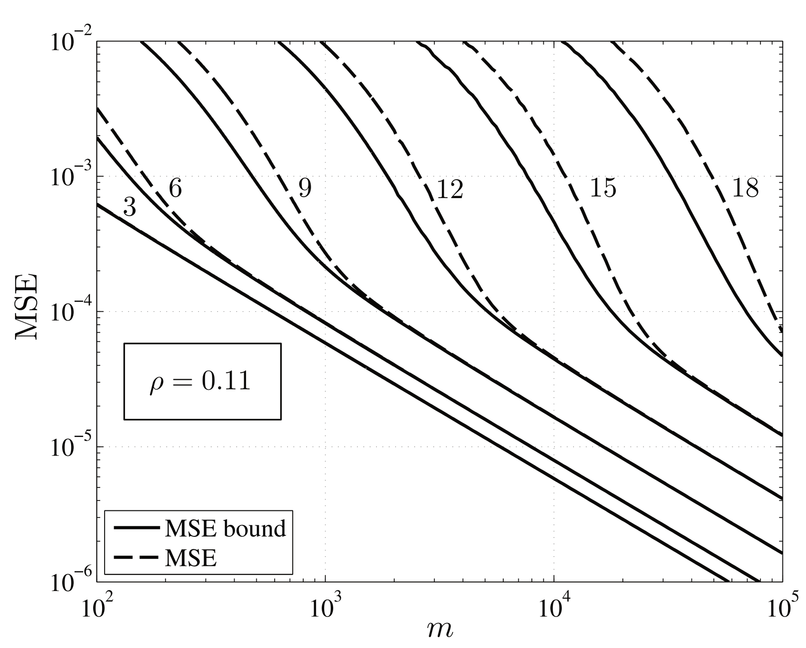
- The analytical mean (6) of the estimator is close to the true parameter ρ.
- The mean (shown as markers) of a simulation of a regular LDPC code matches the analytical result.
- The simulated normalised standard deviation (shown as error bars) is (slightly) larger than the analytical result due to the violation of the independence assumption of the syndrome bits.

6 MEAN SQUARED ERROR COMPARED TO CRAMÉR-RAO LOWER BOUND AS FUNCTIONS OF ERROR RATE AND CHECK DEGREE



- Due to the relatively small number of check nodes there is a relatively large gap between the MSE (8) of the estimator and the Cramér-Rao bound (9).
- A higher check node degree leads to a significant increase of the MSE.

7 MEAN SQUARED ERROR AS FUNCTION OF NUMBER OF CHECK NODES AND CHECK DEGREE



- For small check node degrees d already a relatively small number of check nodes m leads to a small MSE.
- For a large number of check nodes, the curves approach the inverse of the Fisher information.

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