Fundamental Finite Key Limits for Information Reconciliation in Quantum Key Distribution

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Outline

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Quantum Key Distribution (QKD)

- Cryptographic primitive for key agreement
- Two honest parties: Alice and Bob; dishonest party (eavesdropper): Eve.
- Achievement: Alice and Bob create an information-theoretic secure (composable) key.

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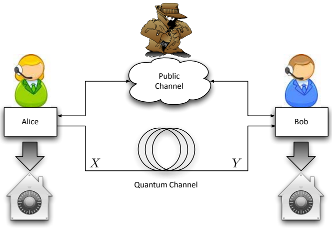
Information-theoretic security (informally)

The success probability of any (active or passive) attack is upper bounded by a (tiny) constant, regardless of the (quantum) computing resources used by the attacker.

QKD protocol steps



Prerequisites:

- Authentic classical channel (Eve can listen)
- Quantum channel (Eve introduces noise while listening)

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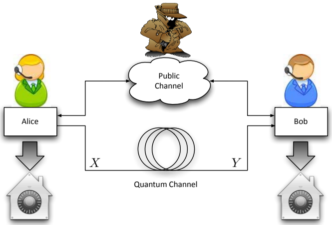

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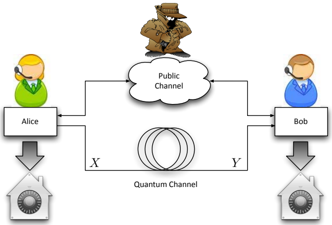

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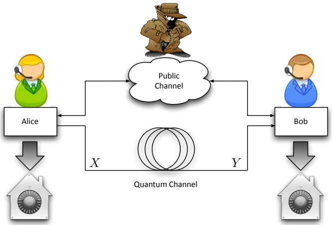

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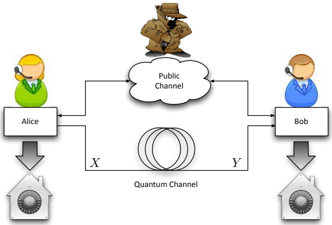

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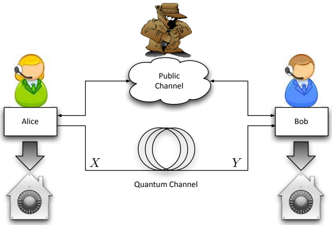

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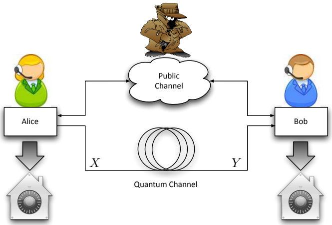

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One Way Information Reconciliation

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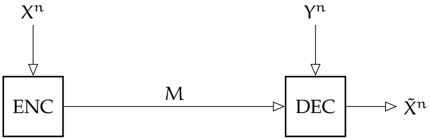

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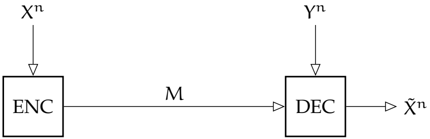

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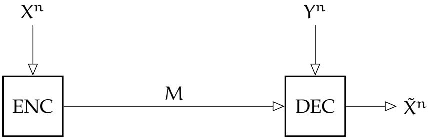

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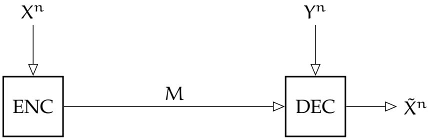

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- Asymptotic limit it is sufficient to send $nH(X|Y)$ bits

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- However, this choice should depend on the distribution P_{XY}, the frame length n, and the frame error rate ε.
- What are the fundamental / practical limits of $\log|\mathcal{M}|$ as a function of P_{XY}, n, and ε?

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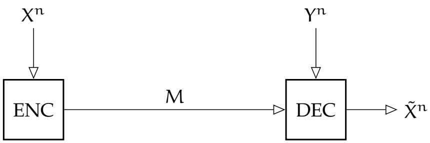
IR / Source coding with side information



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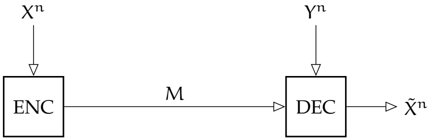

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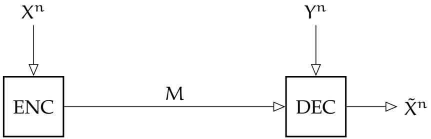

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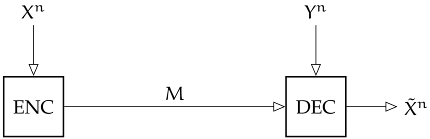

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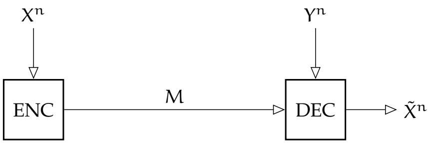

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1. For an arbitrary $(P_{XY})^{\times n}$ we provide the asymptotic expansion up to third order for the converse bound
2. For a special case we provide a non-asymptotic converse bound
3. We compare these bounds to implementations of one-way IR using low-density parity-check codes.

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where $H(X|Y) := \mathrm{Exp}\left[\log \frac{P_Y}{P_{XY}}\right]$ is the conditional entropy, $V(X|Y) := \mathrm{Var}\left[\log \frac{P_Y}{P_{XY}}\right]$ is the conditional entropy variance, and Φ is the cumulative standard normal distribution.

Special Case: Quantum Bit Error Rate Q

P^Q_{XY} results from measurements on a channel with (independent) qber Q:

$$P^Q_X(0) = P^Q_X(1) = P^Q_Y(0) = P^Q_Y(1) = 1/2,$$
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Theorem (Non-asymptotic converse bound for (ε, Q)-correct prot.)

$$\log|\mathcal{M}| \geq nh(Q) + \left(n(1-Q) - F^{-1}\left(\varepsilon\left(1 + 1/\sqrt{n}\right); n, 1-Q\right) - 1\right) \log\frac{1-Q}{Q} - \frac{1}{2}\log n - \log\frac{1}{\varepsilon}.$$

where $F^{-1}(\,\cdot\,; n, p)$ is the inverse of the CDF of the binomial distribution.

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Let $0 < \varepsilon < 1$ and let $0 < Q < \frac{1}{2}$. Then, for large n, any (ε, Q)-correct IR protocol satisfies

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Numerically, this simple bound matches the non-asymptotic bound very well.

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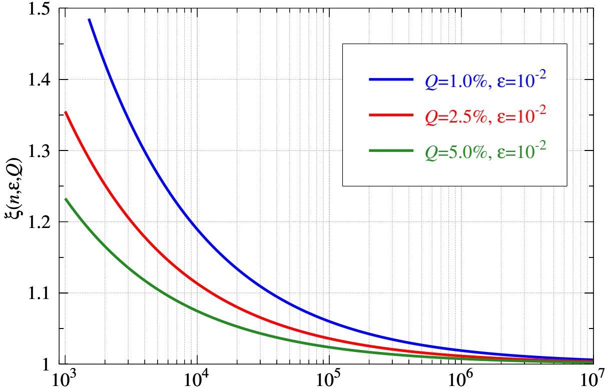

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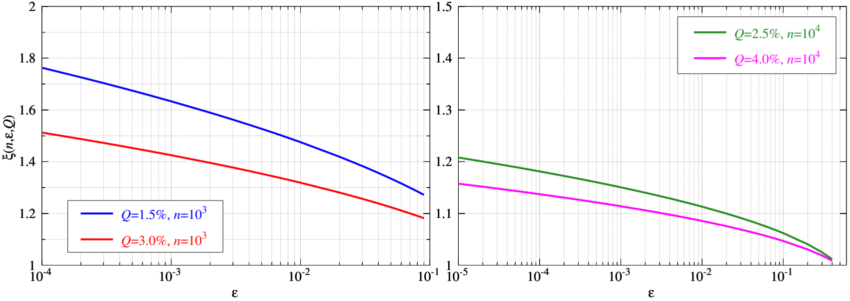

But what about realistic IR codes?

Theoretical Bound

$$\frac{\log|\mathcal{M}|}{nh(Q)} \approx \xi(n,\varepsilon;Q) := 1 + \frac{1}{\sqrt{n}}\frac{\sqrt{v(Q)}}{h(Q)}\Phi^{-1}(1-\varepsilon)$$

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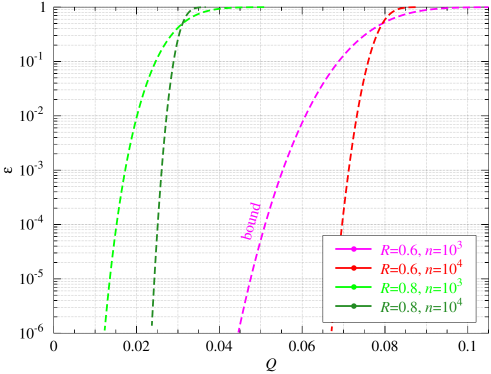

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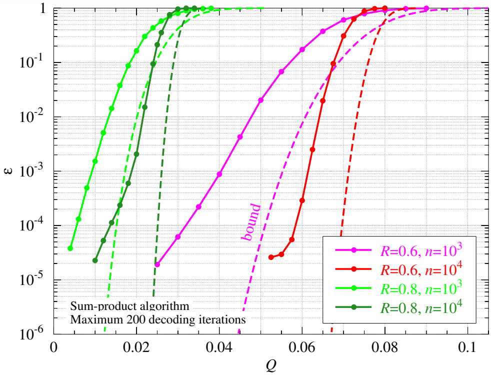

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Conjecture for LDPC codes

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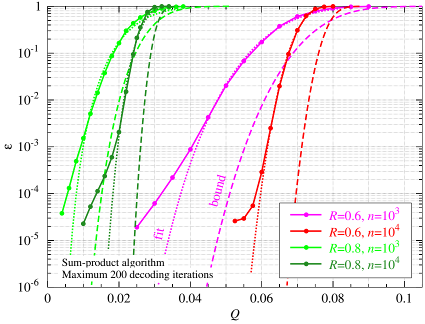

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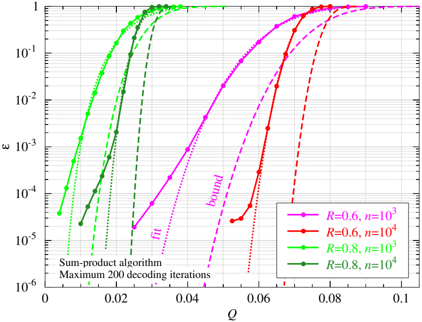
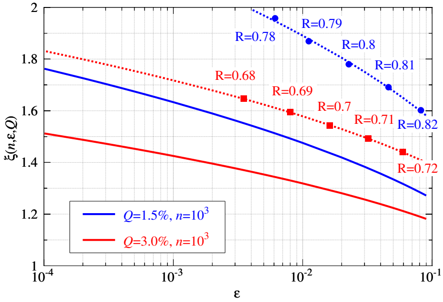
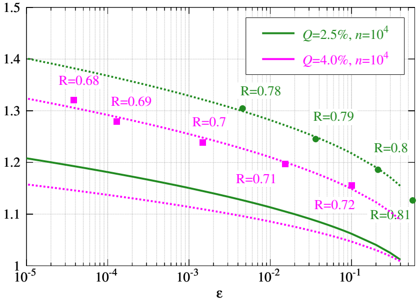


n	$\log\lvert\mathcal{M}\rvert$	ξ_1	ξ_2
10^3	$4 \cdot 10^2$	1.11	1.39
10^3	$3 \cdot 10^2$	1.12	1.45
10^3	$2 \cdot 10^2$	1.13	1.69
10^4	$4 \cdot 10^3$	1.07	1.41
10^4	$3 \cdot 10^3$	1.08	1.44
10^4	$2 \cdot 10^3$	1.11	1.89

But what about realistic IR codes?





n	Q	ξ_1	ξ_2
10^3	0.015	1.16	1.52
10^3	0.030	1.16	1.31

n	Q	ξ_1	ξ_2
10^4	0.025	1.14	1.26
10^4	0.040	1.07	1.58

Outline

Conclusions / Open Questions

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- Commonly used approximation $\log|\mathcal{M}| \approx 1.1nh(Q)$ is often too optimistic for one-way IR
- Numerical simulations for LDPC codes $\rightarrow$ approximation that can be used for the design of QKD systems

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- Behaviour for different code families
- Joint consideration of fundamental limits for finite-length reconciliation and privacy amplification

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THANK YOU!