Security proofs for continuous-variable quantum key distribution

Anthony Leverrier

Inria Paris

QCrypt 2020 - virtual

10 August 2020

Disclaimer

- there won't be any COVID joke, sorry!
- I won't really talk about experimental stuff
- I won't talk about the zillion CVQKD protocols out there, only about a couple that are
  - simple to describe
    AND
  - simple to implement
- the talk might contain controversial[1] statements such as:

"sure, BB84 is a fine protocol, but it's high time we move to CV protocols!"

---

[1] but nothing too provocative! e.g. I won't talk about the quantum Internet

Outline

Discrete versus continuous variables

- BB84 vs CVQKD

State-of-the-art for security proofs

- Gaussian vs discrete modulation of coherent states

Next steps, open questions

- finite size setting, general attacks

Discrete versus continuous variables

Two natural/simple qkd protocols

BB84

- so natural that it would have been discovered eventually (much later?), even without B&B
- distribute copies of $|00\rangle + |11\rangle$
- measure with $\mathbb{1} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1| + |+\rangle\langle +| + |-\rangle\langle -|)$

CVQKD = THE ∞-dim generalization

- distribute copies of $|00\rangle + \lambda|11\rangle + \lambda^2|22\rangle + \cdots + \lambda^k|kk\rangle + \cdots \quad = e^{\lambda \hat{a}^\dagger \hat{b}^\dagger}|\text{vacuum}\rangle$
- measure with $\mathbb{1} = \frac{1}{\pi}\int_{\mathbb{C}} |\alpha\rangle\langle\alpha| d\alpha$, with coherent state $|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{k=0}^{\infty}\frac{\alpha^k}{\sqrt{k!}}|k\rangle \quad = e^{\alpha\hat{a}^\dagger}|\text{vacuum}\rangle$

  a.k.a. *coherent detection*, *heterodyne* measurement, or *double-homodyne* measurement

alternative for CVQKD

- measure the quadratures (homodyne detection) $\Longrightarrow$ the setup of the EPR paper from 1935![2]

[2] formalized much later: Ralph (99), Reid (00), Cerf & al. (01), Grosshans-Grangier (02), Weedbrook & al. (03)...

Two natural/simple qkd protocols

BB84

- so natural that it would have been discovered eventually (much later?), even without B&B
- distribute copies of $|00\rangle + |11\rangle$
- measure with $\mathbb{1} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1| + |+\rangle\langle +| + |-\rangle\langle -|)$

CVQKD = THE ∞-dim generalization

- distribute copies of $|00\rangle + \lambda|11\rangle + \lambda^2|22\rangle + \cdots + \lambda^k|kk\rangle + \cdots \quad = e^{\lambda \hat{a}^\dagger \hat{b}^\dagger}|\text{vacuum}\rangle$
- measure with $\mathbb{1} = \frac{1}{\pi}\int_{\mathbb{C}} |\alpha\rangle\langle\alpha| d\alpha$, with coherent state $|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{k=0}^{\infty}\frac{\alpha^k}{\sqrt{k!}}|k\rangle \quad = e^{\alpha\hat{a}^\dagger}|\text{vacuum}\rangle$

  a.k.a. *coherent detection*, *heterodyne* measurement, or *double-homodyne* measurement

alternative for CVQKD

- measure the quadratures (homodyne detection) $\Longrightarrow$ the setup of the EPR paper from 1935![2]

[2] formalized much later: Ralph (99), Reid (00), Cerf & al. (01), Grosshans-Grangier (02), Weedbrook & al. (03)...

Two natural/simple qkd protocols

BB84

- so natural that it would have been discovered eventually (much later?), even without B&B
- distribute copies of $|00\rangle + |11\rangle$
- measure with $\mathbb{1} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1| + |+\rangle\langle +| + |-\rangle\langle -|)$

CVQKD = THE ∞-dim generalization

- distribute copies of $|00\rangle + \lambda|11\rangle + \lambda^2|22\rangle + \cdots + \lambda^k|kk\rangle + \cdots \quad = e^{\lambda \hat{a}^\dagger \hat{b}^\dagger}|\mathrm{vacuum}\rangle$
- measure with $\mathbb{1} = \frac{1}{\pi}\int_{\mathbb{C}} |\alpha\rangle\langle\alpha| d\alpha$, with coherent state $|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{k=0}^{\infty}\frac{\alpha^k}{\sqrt{k!}}|k\rangle \quad = e^{\alpha\hat{a}^\dagger}|\mathrm{vacuum}\rangle$

  a.k.a. *coherent detection*, *heterodyne* measurement, or *double-homodyne* measurement

alternative for CVQKD

- measure the quadratures (homodyne detection) $\implies$ the setup of the EPR paper from 1935![2]

[2] formalized much later: Ralph (99), Reid (00), Cerf & al. (01), Grosshans-Grangier (02), Weedbrook & al. (03)...

Theory vs practice

BB84 in practice: NOT SO SIMPLE!

- single photons are usually prepared via $|00\rangle + \lambda|11\rangle + \lambda^2|22\rangle + \cdots + \lambda^k|kk\rangle + \cdots$ and heralding
- experimentally-friendlier version of BB84 relies on (phase-randomized) coherent states

$\Longrightarrow$ same states as in CVQKD! requires to ~~tweak~~ completely redo the analysis (multi-photon pulses)

- photon counters hard to implement replaced by threshold detectors

$\Longrightarrow$ infinite-dimensional Fock space, same as CVQKD!

CVQKD: pretty much as advertised

- same states, same measurement as specified (modulo a finite precision issue)
- P&M version: Alice prepares $|\alpha\rangle$ with $\alpha \sim \mathcal{N}_{\mathbb{C}}(0, \sigma^2)$ (or α from finite set)
- implementations today closely match the original protocols

my personal (provocative) view:
BB84 was nice to launch the field of quantum crypto, but the future belongs to CV!

Theory vs practice

BB84 in practice: NOT SO SIMPLE!

- single photons are usually prepared via $|00\rangle + \lambda|11\rangle + \lambda^2|22\rangle + \cdots + \lambda^k|kk\rangle + \cdots$ and heralding
- experimentally-friendlier version of BB84 relies on (phase-randomized) coherent states

$\Longrightarrow$ same states as in CVQKD! requires to ~~tweak~~ completely redo the analysis (multi-photon pulses)

- photon counters hard to implement replaced by threshold detectors

$\Longrightarrow$ infinite-dimensional Fock space, same as CVQKD!

CVQKD: pretty much as advertised

- same states, same measurement as specified (modulo a finite precision issue)
- P&M version: Alice prepares $|\alpha\rangle$ with $\alpha \sim \mathcal{N}_{\mathbb{C}}(0, \sigma^2)$ (or α from finite set)
- implementations today closely match the original protocols

my personal (provocative) view:
BB84 was nice to launch the field of quantum crypto, but the future belongs to CV!

Theory vs practice

BB84 in practice: NOT SO SIMPLE!

- single photons are usually prepared via $|00\rangle + \lambda|11\rangle + \lambda^2|22\rangle + \cdots + \lambda^k|kk\rangle + \cdots$ and heralding
- experimentally-friendlier version of BB84 relies on (phase-randomized) coherent states

$\Longrightarrow$ same states as in CVQKD! requires to ~~tweak~~ completely redo the analysis (multi-photon pulses)

- photon counters hard to implement replaced by threshold detectors

$\Longrightarrow$ infinite-dimensional Fock space, same as CVQKD!

CVQKD: pretty much as advertised

- same states, same measurement as specified (modulo a finite precision issue)
- P&M version: Alice prepares $|\alpha\rangle$ with $\alpha \sim \mathcal{N}_{\mathbb{C}}(0, \sigma^2)$ (or α from finite set)
- implementations today closely match the original protocols

my personal (provocative) view:
BB84 was nice to launch the field of quantum crypto, but the future belongs to CV!

ok... are there any drawbacks to CVQKD?

~~of course not!~~

More challenging theory[3]

- ∞ dimension (same is kind of true for implementations of DVQKD)
- continuous-valued AND unbounded measurement operators
- quality of the correlations measured via *covariance matrix (unbounded)*, not QBER or CHSH score

$\Longrightarrow$ conceptual difficulties, but rather *clean problems*

Experimental performance: seems less robust to loss than DV

- losses are filtered out for DV: discard the no-click events[4]
- all pulses are there for CV, but noisier $\Longrightarrow$ harder to estimate the channel parameters precisely
- very large blocks required for long distance

[3] modern DVQKD protocols are also very complex!

[4] modulo some assumptions on the detectors (as demonstrated by Vadim Makarov!)

ok... are there any drawbacks to CVQKD?

~~of course not!~~

More challenging theory[3]

- ∞ dimension (same is kind of true for implementations of DVQKD)
- continuous-valued AND unbounded measurement operators
- quality of the correlations measured via *covariance matrix (unbounded)*, not QBER or CHSH score

  $\Longrightarrow$ conceptual difficulties, but rather *clean problems*

Experimental performance: seems less robust to loss than DV

- losses are filtered out for DV: discard the no-click events[4]
- all pulses are there for CV, but noisier $\Longrightarrow$ harder to estimate the channel parameters precisely
- very large blocks required for long distance

[3] modern DVQKD protocols are also very complex!

[4] modulo some assumptions on the detectors (as demonstrated by Vadim Makarov!)

P&M version of CVQKD

- Alice sends $|\alpha_1\rangle, \cdots |\alpha_n\rangle$
  - α_k either Gaussian variable or element from a finite set (e.g. $\{\pm\alpha, \pm i\alpha\}$)
- Bob measures with heterodyne detection: gets $\beta_1, \cdots, \beta_n \in \mathbb{C}$.
  - typical model: $\beta = t\alpha + \gamma$ with fixed *attenuation* t and Gaussian noise $\gamma \sim \mathcal{N}_{\mathbb{C}}(0, 1 + t^2\xi)$
  - $t \sim 0.1$ at 100km
  - ξ is the *excess noise*: $10^{-3} - 10^{-2}$ in implementations $\implies$ hard to mesure precisely
- classical postprocessing (essentially identical to DV)
  - key map: from Bob's data (reverse reconcilation[5])

$$\beta_1, \cdots \beta_n \to x_1, \cdots x_N \in \{0, 1\}$$

  - parameter estimation: covariance matrix of α, β
    (informally, want to estimate t, ξ) $\implies$ *the most challenging part*
  - privacy amplification

[5] actually same for BB84 due to discarding no-click events

P&M version of CVQKD

- Alice sends $|\alpha_1\rangle, \cdots |\alpha_n\rangle$
  - α_k either Gaussian variable or element from a finite set (e.g. $\{\pm\alpha, \pm i\alpha\}$)
- Bob measures with heterodyne detection: gets $\beta_1, \cdots, \beta_n \in \mathbb{C}$.
  - typical model: $\beta = t\alpha + \gamma$ with fixed *attenuation* t and Gaussian noise $\gamma \sim \mathcal{N}_{\mathbb{C}}(0, 1 + t^2\xi)$
  - $t \sim 0.1$ at 100km
  - ξ is the *excess noise*: $10^{-3} - 10^{-2}$ in implementations $\implies$ hard to mesure precisely
- classical postprocessing (essentially identical to DV)
  - key map: from Bob's data (reverse reconcilation[5])

$$\beta_1, \cdots \beta_n \to x_1, \cdots x_N \in \{0, 1\}$$

  - parameter estimation: covariance matrix of α, β
    (informally, want to estimate t, ξ) $\implies$ *the most challenging part*
  - privacy amplification

[5] actually same for BB84 due to discarding no-click events

P&M version of CVQKD

- Alice sends $|\alpha_1\rangle, \cdots |\alpha_n\rangle$
  - α_k either Gaussian variable or element from a finite set (e.g. $\{\pm\alpha, \pm i\alpha\}$)
- Bob measures with heterodyne detection: gets $\beta_1, \cdots, \beta_n \in \mathbb{C}$.
  - typical model: $\beta = t\alpha + \gamma$ with fixed *attenuation* t and Gaussian noise $\gamma \sim \mathcal{N}_{\mathbb{C}}(0, 1 + t^2\xi)$
  - $t \sim 0.1$ at 100km
  - ξ is the *excess noise*: $10^{-3} - 10^{-2}$ in implementations $\implies$ hard to mesure precisely
- classical postprocessing (essentially identical to DV)
  - key map: from Bob's data (reverse reconcilation[5])

$$\beta_1, \cdots \beta_n \to x_1, \cdots x_N \in \{0, 1\}$$

  - parameter estimation: covariance matrix of α, β
    (informally, want to estimate t, ξ) $\implies$ *the most challenging part*
  - privacy amplification

[5] actually same for BB84 due to discarding no-click events

CV or DV?

- photons live in ∞-dimensional Fock space: why encode information on some qubit space?
- the simplest states to prepare are coherent (= Gaussian) states! (already used in telecom industry)
- coherent (heterodyne) detection is needed for the whole telecom industry: huge incentives!
- more natural/efficient to encode information in phase-space: continuous variables!
- what about DI / MDI /TF QKD? those don't really work with CV... Well, they're only needed because we don't quite know how to implement vanilla BB84 :-)

$\Longrightarrow$ *qubits are good for computing, less for communicating* *classical information*

Outline

Discrete versus continuous variables

- BB84 vs CVQKD

State-of-the-art for security proofs

- Gaussian vs discrete modulation of coherent states

Next steps, open questions

- finite size setting, general attacks

State-of-the-art for security proofs

QKD as a tomography problem

Goal

get sufficient correlations between A and B to upper bound on Eve's information about $\vec{x}$:

- composable security: $H^{\varepsilon}_{\mathsf{min}}(X_1, \cdots, X_N|E)_{\rho^{(n)}_{AXE}}$
- asymptotic bound[6]: $H(X_1|E)_{\rho_{AXE}}$ (single channel use)

major difficulty already for collective attacks in the asymptotic limit: ρ_{AXE} is a pure

- 4-qubit state for BB84: 16 parameters
- 4-mode state in $\mathsf{Span}(|i, j, k, \ell\rangle \; : \; i, j, k, \ell \in \mathbb{N})$ for CVQKD; even truncating the Fock space to 10 photons/mode gives more than 10^4 parameters

One (only?) useful tool: von Neumann entropy maximized by Gaussian states $S(\rho) \leq S(\rho_G)$

QKD version: $\chi(\beta, E)_\rho \leq \chi(\beta, E)_{\rho_G}$ (ρ_G the Gaussian state with same covariance matrix as ρ)

$\Longrightarrow$ asymptotic security against collective attacks for protocols with Gaussian modulation

[Wolf, Giedke, Cirac PRL 2005] [Garcia-Patron, Cerf PRL 2006] [Navascues, Grosshans, Acin PRL 2006]

[6] for "nice" protocols

QKD as a tomography problem

Goal

get sufficient correlations between A and B to upper bound on Eve's information about $\vec{x}$:

- composable security: $H^{\varepsilon}_{\min}(X_1, \cdots, X_N|E)_{\rho^{(n)}_{AXE}}$
- asymptotic bound[6]: $H(X_1|E)_{\rho_{AXE}}$ (single channel use)

major difficulty already for collective attacks in the asymptotic limit: ρ_{AXE} is a pure

- 4-qubit state for BB84: 16 parameters
- 4-mode state in $\mathrm{Span}(|i, j, k, \ell\rangle \ : \ i, j, k, \ell \in \mathbb{N})$ for CVQKD; even truncating the Fock space to 10 photons/mode gives more than 10^4 parameters

One (only?) useful tool: von Neumann entropy maximized by Gaussian states $S(\rho) \leq S(\rho_G)$

QKD version: $\chi(\beta, E)_\rho \leq \chi(\beta, E)_{\rho_G}$ (ρ_G the Gaussian state with same covariance matrix as ρ)

$\Longrightarrow$ asymptotic security against collective attacks for protocols with Gaussian modulation

[Wolf, Giedke, Cirac PRL 2005] [Garcia-Patron, Cerf PRL 2006] [Navascues, Grosshans, Acin PRL 2006]

[6] for "nice" protocols

Last few years

- Gaussian modulation: essentially solved!
- discrete modulation: still very open, and somewhat pressing issue!

Gaussian modulation: $\alpha \sim \mathcal{N}_{\mathbb{C}}(0, \sigma^2)$

2 approaches to prove security against general attacks:

Entropic uncertainty relation [Furrer & al. PRL 2012]

- discretize $\implies X_\delta, P_\delta$
- $H^\varepsilon_{\min}(X_\delta|E)_{\rho^n} + H^\varepsilon_{\max}(P_\delta|B)_{\rho^n} \geq -\log \frac{\delta^2}{2\pi} S_0^{(1)}(1, \frac{\delta^2}{4})^2$

but protocol requires squeezed states, bound not believed to be tight

*Gaussian de Finetti [**AL** PRL 2017]*

crucial fact: protocol is symmetric wrt $U(n)$ (instead of S_n for BB84) $\implies$ stronger de Finetti

1. symmetrize in phase-space $\implies$ restrict to $\rho^n = \rho_G^{\otimes n}$
2. equipartition property: $H^\varepsilon_{\min}(X_\delta|E)_{\rho_G^{\otimes n}} \approx nH(X_\delta|E)_{\rho_G}$
3. $H(X_\delta|E)_{\rho_{\text{Gauss}}} = H(X_\delta) - \chi(X_\delta; E)_{\rho_G}$
4. estimation of CM $\implies$ upper bound on $\chi(X_\delta; E)_{\rho_G}$

missing element: finite precision of measurements

Discrete modulation

Lorenz & al. (2004), Namiki, Hirano (2006), Zhao & al. (2009), AL, Grangier (2009), Sych, Leuchs (2010), Bradler, Weedbrook (2017)...

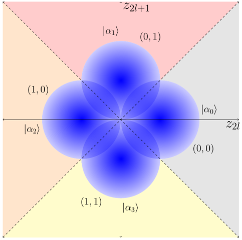
- easier to implement: same as coherent telecom industry
- better for error correction

$\Longrightarrow$ huge interest from industry, H2020 CiViQ

theory is more complicated

- EUR don't help (coherent states)
- U(n)-symmetry is broken $\Longrightarrow$ no Gaussian de Finetti, unclear how to perform PE
- non-Gaussian E-B protocol: pb for bounding vN entropy

$\Longrightarrow$ even asymptotic collective attacks are nontrivial!



Very recent finite-size analysis of a 2-state protocol [*Matsuura & al.* arXiv : 2006.04661]

- mapping to a qubit protocol, but 2 states aren't sufficient to get very good performance
- unclear how to extend to 4 states or more

Two recent results on the 4-state protocol

asymptotic security for collective attacks, assuming channel parameters are known

main idea: convex optimization to bound Holevo information / conditional vN entropy

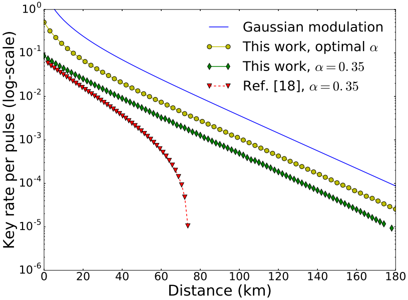
*Ghorai, Grangier, Diamanti, **AL** PRX 19*

- SDP to bound $f(\rho) = \mathrm{tr}((\hat{q}_A \hat{q}_B - \hat{p}_A \hat{p}_B)\rho)$
  $+$ Gaussian optimality
- pro: simple optimization, can be extended to larger constellations
- con: bounds are not tight

Lin, Upadhyaya, Lütkenhaus PRX 19: better (for now)

- SDP to bound $H(X|E)$ directly: $f(\rho) = D\big(\mathcal{G}(\rho)||\mathcal{Z}[\mathcal{G}(\rho)]\big)$
- pro: much tighter key rate
- con: nonlinear objective function, optimization more involved (follows techniques from Coles & al. Nat. Comm. 16)

$$\begin{aligned} \text{minimize} \quad & f(\rho) \\ \text{subject to} \quad & \rho \succeq 0 \\ & \mathrm{tr}(\rho\, \hat{O}_{PM}) = o_{PM} \\ & \mathrm{tr}(\rho) = 1 \end{aligned}$$



(from Lin & al. 2019)

Limitations of these 2 works

- only numerical results
- the true SDP cannot be solved directly because of ∞ dim $\implies$ heuristic truncation of Hilbert space
  - seems ok, but no proof
  - see recent work by Upadhyaya & al. (poster # 92)
- only deal with ideal detection
  - rather easy to patch with approach from Ghorai & al. (still won't be tight)
  - harder for Lin & al. (see poster # 28)
- parameter estimation is ignored!
- what about larger constellations? the results from Ghorai & al. should get much tighter

Outline

Discrete versus continuous variables

- BB84 vs CVQKD

State-of-the-art for security proofs

- Gaussian vs discrete modulation of coherent states

Next steps, open questions

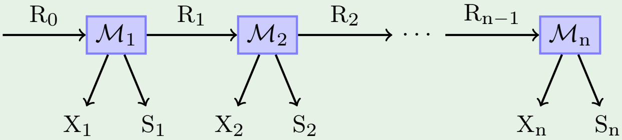
- finite size setting, general attacks

Next steps, open questions

Going further: security against general attacks, finite-size setting?

a potential approach: the entropy accumulation theorem [Dupuis, Fawzi, Renner 2016]

- gives tight bounds for DV QKD
- successfully applied to device-independent QKD [Arnon-Friedman & al. 2018]



- $H^{\varepsilon}_{\min}(X_1 \cdots X_n | ES^n)_{\rho^n} \geq n \min_\sigma H(X_1|ES_1)_\sigma - O(\sqrt{n})$

difficulties to adapt EAT to CV:

- requires some test. Seems much harder to define than for DV: should be related to covariance matrix, but not clear how
- test depends on some unbounded continuous outcome

The real difficulty: unbounded variables

Given $x_1, \ldots, x_n \in \mathbb{R}$ i.i.d. from unknown distribution with $\langle x \rangle = 0$, estimate $\langle x^2 \rangle$

random sampling doesn't work, e.g.,

$$x_i = \begin{cases} 0 & \text{with prob } 1 - \varepsilon \\ \pm C & \text{with prob } \varepsilon/2 \end{cases}$$

$\implies$ $\langle x^2 \rangle = C^2 \varepsilon$ but requires to sample a fraction $\geq 1 - \varepsilon$

Solution: rotational symmetry

- apply random $R \in O(n)$ to $\vec{x}$: $\quad \vec{x} \to R\vec{x}$,
- sample first k coordinates
- concentration of measure gives tight bounds

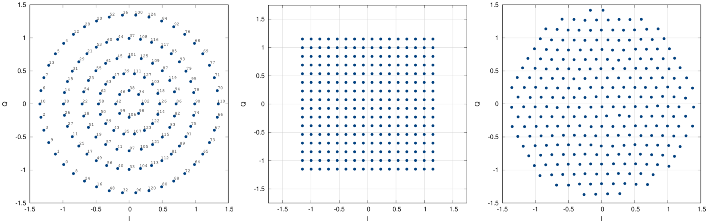
$\implies$ bound on CM for protocols with Gaussian modulation $\implies$ security against collective attacks [AL PRL 2015]

Unclear how to perform PE for discrete modulation at the moment...

unless restricted attack setting (e.g. Papanastasiou, Pirandola arXiv:1912.11418)

Optimal constellation?

- infinitely precise Gaussian modulation isn't physical $\Longrightarrow$ finite constellations
- 2 or 3 states aren't enough to get good performance
- 4 states are ok, but larger constellations should allow for larger variance
  - improved asymptotics: key rate $\times 10$?
  - better for PE, for finite-size
  - "easy" for telecom industry



- previous results should extend there but unclear how tractable will be the numerics
- very large constellations might allow for continuity-type arguments (Kaur, Guha, Wilde arXiv:1901.10099)

Conclusion and perspectives

Conclusion and perspectives

- CV are well-suited to large-scale deployment of QKD:

  compatible with telecom industry standards

- security is quite involved (infinite dimension, unbounded variables, discretization, truncation...) but *not more than for modern DVQKD protocols*, and with *cleaner problems*?

challenges for theorists

- is it possible to apply entropy accumulation?
- how to perform parameter estimation without rotation symmetry? (for discrete modulation)
- what is better: 4 states or large constellations?

Thanks!

CiViQ